

ON \mathbb{T} -SHY SETS IN RADON METRIC GROUPS

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Abstract

Let \mathbf{G} be a complete metric group and \mathbb{T} be such a subgroup of its Borel automorphisms group $\mathbb{A}(\mathbf{G})$, which contains all left and right shifts of the \mathbf{G} . We introduce notions of \mathbb{T} -cm and \mathbb{T} -shy sets and demonstrate that they constitute a σ -ideal and coincide in Radon metric groups. This result extends main results established in [6], [13], [20]. For a Borel probability measure μ in a Polish group \mathbf{G} , we construct the generator $\mathbf{G}_{(\mathbb{T}, \mu)}$ of (\mathbb{T}, μ) -shy sets which is quasi-finite, whenever the μ is a \mathbb{T} -quasi-generator. Using Okazaki-Takahashi result [23], we prove that, if a group of admissible translations (in the sense of quasi-invariance) for a Borel probability μ in a Polish topological vector space \mathbb{V} is thick, then the generator $\mathbf{G}_{(\pi(\mathbb{V}), \mu)}$ of $(\pi(\mathbb{V}), \mu)$ -shy (equivalently, shy) sets is quasi-finite. For such a Borel measure μ , we construct a quasi-finite semi-finite

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translation-invariant Borel measure $\mathbf{G}_{(\pi(\mathbb{V}), \nu)}$, which is equivalent to the generator $\mathbf{G}_{(\pi(\mathbb{V}), \mu)}$. Also, we show that Okazaki's dichotomy [22] is not valid for generators of \mathbb{T} -shy sets in \mathbb{V} . By using technique of $\pi(B)$ -cm and $\pi(B)$ -shy sets, we solve negatively a certain problem posed by Fremlin in [9].

1. Introduction

In [30], for a Polish topological vector space \mathbb{V} has been introduced the notion of a generator of shy sets μ , which is such a Borel measure in \mathbb{V} that a condition $\bar{\mu}(X) = 0$ implies that X is Haar null(or shy) in the sense [6], where $\bar{\mu}$ denotes a usual completion of the μ . Here has been demonstrated that the class of generators of shy sets in \mathbb{V} contains specific Borel measures, which naturally generate implicitly introduced subclasses of shy sets (see, for example, [3], [6], [8], [13], [19], [36], [37], [38]). Moreover, such measures (unlike σ -finite Borel measures) possess many interesting, sometimes unexpected, geometric properties (see, for example, [27], [30]).

Some applications of generators of shy sets in infinite-dimensional analysis have been considered in [28], [29], [30], [31]. For example, for a normal Hermitian operator A with a convergent trace $\text{Tr}(A)$, an example of a quasi-finite translation-invariant Borel measure μ_A in the infinite-dimensional separable Hilbert space ℓ_2 has been constructed in [28] such that $\mu_A(e^{tA}(D_0)) = e^{t\text{Tr}(A)}\mu_A(D_0)$, for an arbitrary Borel subset D_0 in ℓ_2 . In [31], the class \mathbb{A} of all infinite-dimensional diagonal matrices and the class \mathbb{F} of all infinite collections of continuous impulses have been described such that Mankiewicz generator \mathbf{G}_M [30] preserves the phase flow (in $\mathbb{R}^{\mathbb{N}}$) defined by the non-homogeneous differential equation $\frac{d\Psi}{dt} = A \times \Psi + f$, for $A \in \mathbb{A}$, $f \in \mathbb{F}$. An analogous question has been studied for Preiss-Tišer generators in $\mathbb{R}^{\mathbb{N}}$ [30]. In [33], it has been shown that the class of generators of shy sets in an abelian Polish group G is non-empty,

if G contains any uncountable locally compact Hausdorff topological subgroup. This result (see [33], Theorem 2.1) extends a certain result early obtained for Polish topological vector spaces in [30] (see Theorem 2.1, p. 238). For a Borel probability measure μ in a Polish topological vector space \mathbb{V} , has been constructed a generator of shy sets \mathbf{G}_μ such that a subclass of shy sets generated by the measure μ coincides with the class of $\overline{\mathbf{G}_\mu}$ -zero sets, where $\overline{\mathbf{G}_\mu}$ denotes a usual completion of the generator \mathbf{G}_μ (see Theorem 3.1, p. 245).

A suitable extension of the property of being of Haar measure zero (or shy) in abelian Polish groups to all non-abelian Polish groups has been given in Mycielski [20]. Like this approach, the notion of generators of shy sets introduced for Polish topological vector spaces [30] has been extended to all Polish groups in [33]. In this paper, an example of a two sided invariant generator of shy sets has been constructed in the product of unimodular Polish groups. For a compact set K in the product of a countable family of non-compact unimodular Polish groups, has been constructed a two sided invariant generator of shy sets μ such that $\mu(K) = 0$. Also, here is extended Dougherty's criterion of shyness in the \mathbf{G} being the product of a countable family of unimodular Polish groups (that are not compact). In addition, it has been shown that if \mathbf{G} has an invariant metric, then every generator of shy sets in \mathbf{G} is non- σ -finite.

The purpose of the present paper is to introduce a new concept of small (the so called, of \mathbb{T} -cm and \mathbb{T} -shy) sets in a complete metric group and to develop the corresponding theory.

The paper is organized as follows.

In Section 2, we study relation between invariance and quasi-invariance for Borel probability measures in Radon metric spaces.

In Section 3, we introduce a notion of \mathbb{T} -shy sets in a Radon metric group and show that this class constitute a σ -ideal. We discuss whether this result covers main results established in [6], [13], [20].

An existence of a quasi-finite generator of (\mathbb{T}, μ) -shy sets in a Radon metric group \mathbf{G} is studied in Section 4. We demonstrate that such a Borel measure exists, if the μ is \mathbb{T} -quasi-generator (see Theorem 3.2). This result extends a certain result early established for Polish topological vector spaces (see [30], Theorem 3.1, p. 245).

In Section 5, by using Okazaki-Takahashi result [23], we prove that if a group of admissible translations (in the sense of quasi-invariance) for a Borel probability measure μ with domain in a Polish topological vector space \mathbb{V} is thick, then the generator of $(\pi(\mathbb{V}), \mu)$ -shy sets $\mathbf{G}_{(\pi(\mathbb{V}), \mu)}$ constructed in [30] is quasi-finite. For such a Borel measure μ , we construct a quasi-finite semi-finite translation-invariant Borel measure $\mathbf{G}_{(\pi(\mathbb{V}), \nu)}$, which is equivalent to the generator $\mathbf{G}_{(\pi(\mathbb{V}), \mu)}$. Also, we show that Okazaki's dichotomy [22] is not valid for generators of shy sets in Radon metric groups.

In Section 6, we consider an example of a quasi-finite generator of \mathcal{G}_α -shy sets in the abelian Polish group \mathbb{R}^∞ , where \mathcal{G}_α denotes a group of Borel automorphisms of the \mathbb{R}^∞ generated by all α -permutations and shifts of the basic space.

In Section 7, by using the technique of $\pi(B)$ -cm and $\pi(B)$ -shy sets, we solve negatively a certain problem posed by Fremlin in [9].

2. Relation between Invariance and Quasi-Invariance for Borel Probability Measures in Radon Metric Spaces

Recall that a complete metric space \mathbb{E} is called a *Radon metric space*, if every Borel probability measure is tight, i.e., for every $\epsilon > 0$, there is a compact set C_ϵ such that $\mu(C_\epsilon) > \mu(E) - \epsilon$.

Let μ be a diffused probability Borel measure defined on the Radon metric space \mathbb{E} and let $\mathcal{B}(\mathbb{E})$ be a Borel σ -algebra of subsets of \mathbb{E} .

Definition 2.1. A Borel automorphism $\mathbf{g} : \mathbb{E} \rightarrow \mathbb{E}$ is called admissible in the sense of quasi-invariance for the measure μ , if

$$(\forall \mathbf{X})(\mathbf{X} \in \mathcal{B}(\mathbb{E}) \rightarrow (\mu(\mathbf{X}) = \mathbf{0} \Leftrightarrow \mu(\mathbf{g}(\mathbf{X})) = \mathbf{0})).$$

Remark 2.1. Let \mathbb{Q}_μ denotes a set of all admissible automorphisms in the sense of quasi-invariance for the measure μ . It is obvious that the \mathbb{Q}_μ is a group with respect to the usual composition operation “ \circ ”.

Definition 2.2. A Borel automorphism $g : \mathbb{E} \rightarrow \mathbb{E}$ is called admissible in the sense of invariance for the measure μ , if

$$(\forall \mathbf{X})(\mathbf{X} \in \mathcal{B}(\mathbb{E}) \rightarrow \mu(\mathbf{X}) = \mu(\mathbf{g}(\mathbf{X}))).$$

Remark 2.2. Let \mathbb{I}_μ denotes a set of all admissible automorphisms in the sense of invariance for the measure μ . Like the \mathbb{Q}_μ , the \mathbb{I}_μ also is a group with respect to the usual composition operation “ \circ ”.

In the sequel, we need some auxiliary lemmas.

Lemma 2.1. *Let \mathbb{E}_1 and \mathbb{E}_2 be Polish topological spaces. Let μ_1 and μ_2 be probability diffused Borel measures defined on \mathbb{E}_1 and \mathbb{E}_2 , respectively. Then, there exists a Borel isomorphism $\varphi : (\mathbb{E}_1, \mathcal{B}(\mathbb{E}_1)) \rightarrow (\mathbb{E}_2, \mathcal{B}(\mathbb{E}_2))$ such that*

$$\mu_1(\mathbf{X}) = \mu_2(\varphi(\mathbf{X})),$$

for every $\mathbf{X} \in \mathcal{B}(\mathbb{E}_1)$.

The proof of Lemma 2.1 can be found in [5].

The next auxiliary proposition plays the key role in our further consideration.

Lemma 2.2 (Ulam). *Let (\mathbb{E}, ρ) be a Radon metric space. Let μ be a σ -finite Borel measure defined on \mathbb{E} . Then, there exists a closed separable subspace $\mathbb{E}(\mu)$ of the \mathbb{E} such that*

$$\mu(\mathbb{E} \setminus \mathbb{E}(\mu)) = 0.$$

Remark 2.3. We remind the reader that a cardinal number α is real-valued measurable, if there exists a continuous probability measure defined on the class of all subsets of some set of cardinality α . In connection with Lemma 2.2, we must also recall that an arbitrary complete metric space (\mathbb{E}, ρ) , whose topological weight is not a real-valued measurable cardinal, is a Radon metric space (cf. [17], p. 48, Theorem 7). Moreover, the following conditions are equivalent:

- (a) an arbitrary complete metric space is a Radon space;
- (b) there does not exist a real-valued measurable cardinal.

Theorem 2.1. *Let μ be a diffused probability Borel measure defined on the Radon metric space \mathbb{E} . Then*

$$\mathbb{I}_\mu \subset \mathbb{Q}_\mu \text{ \& } \mathbb{Q}_\mu \setminus \mathbb{I}_\mu \neq \emptyset.$$

Proof. Let $\mathbf{g} \in \mathbb{I}_\mu$. Then

$$(\forall \mathbf{X})(\mathbf{X} \in \mathcal{B}(\mathbb{E}) \rightarrow \mu(\mathbf{g}(\mathbf{X})) = \mu(\mathbf{X})).$$

It follows that $\mu(\mathbf{X}) = \mathbf{0}$ iff $\mu(\mathbf{g}(\mathbf{X})) = \mathbf{0}$. The latter relation means that $\mathbf{g} \in \mathbb{Q}_\mu$. Now, we have to show that $\mathbb{Q}_\mu \setminus \mathbb{I}_\mu \neq \emptyset$.

By Lemma 2.2, there exists a closed separable subspace $\mathbb{E}(\mu)$ of the \mathbb{E} such that

$$\mu(\mathbb{E} \setminus \mathbb{E}(\mu)) = 0.$$

Note that the closed separable subset $\mathbb{E}(\mu)$ of the complete metric space (\mathbb{E}, ρ) with the restriction ρ_1 of the metric ρ to the set $\mathbb{E}(\mu) \times \mathbb{E}(\mu)$ is a Polish space.

Let γ be a standard Gaussian measure on \mathbb{R} .

By Lemma 2.1, there is a Borel isomorphism

$$\Phi : \mathbb{E}(\mu) \rightarrow \mathbb{R},$$

such that $(\forall \mathbf{X})(\mathbf{X} \in \mathcal{B}(\mathbb{E}(\mu)) \rightarrow \mu(\mathbf{X}) = \gamma(\Phi(\mathbf{X})))$.

Let us consider a group \mathbf{G}_0 of Borel automorphisms of the $\mathbb{E}(\mu)$ defined by

$$\mathbf{G}_0 = \{\Phi \circ \Phi_{\mathbf{h}} \circ \Phi^{-1} : \mathbf{h} \in \mathbb{R}\},$$

where $\Phi_{\mathbf{h}}$ is a shift in \mathbb{R} defined by $(\forall \mathbf{x})(\mathbf{x} \in \mathbb{R} \rightarrow \Phi_{\mathbf{h}}(\mathbf{x}) = \mathbf{x} + \mathbf{h})$.

For $h \in \mathbb{R}$, we set $g_h(x) = \Phi \circ \Phi_h \circ \Phi^{-1}$, for $x \in \mathbb{E}(\mu)$ and $g_h(x) = x$, for $x \in \mathbb{E} \setminus \mathbb{E}(\mu)$.

For $X \in \mathcal{B}(\mathbb{E})$, we have

$$\begin{aligned} \mu(X) = 0 &\Leftrightarrow \mu(X \cap \mathbb{E}(\mu)) = 0 \Leftrightarrow \gamma(\Phi(X \cap \mathbb{E}(\mu))) = 0 \Leftrightarrow \gamma(\Phi_h(\Phi(X \cap \mathbb{E}(\mu)))) \\ &= 0 \Leftrightarrow \gamma(\Phi^{-1}(\Phi_h(\Phi(X \cap \mathbb{E}(\mu)))) = 0 \Leftrightarrow \mu(\Phi \circ \Phi_h \circ \Phi^{-1}(X \cap \mathbb{E}(\mu))) \\ &= 0 \Leftrightarrow \mu(\Phi \circ \Phi_h \circ \Phi^{-1}(X \cap \mathbb{E}(\mu))) + \mu(\Phi \circ \Phi_h \circ \Phi^{-1}(X) \cap \mathbb{E} \setminus \mathbb{E}(\mu)) \\ &= 0 \Leftrightarrow \mu(\Phi \circ \Phi_h \circ \Phi^{-1}(X) \cap \mathbb{E}) + \mu(\Phi \circ \Phi_h \circ \Phi^{-1}(X) \cap (\mathbb{E} \setminus \mathbb{E}(\mu))) \\ &= 0 \Leftrightarrow \mu(g_h(X)) = 0. \end{aligned}$$

The latter relation means that $\mathbf{G}_0 \subset \mathbb{Q}_\mu$.

Let $\mathbf{Y}_0 \in \mathcal{B}(\mathbb{R})$ and $\mathbf{h}_0 \in \mathbb{R}$ such that

$$\gamma(\mathbf{Y}_0 + \mathbf{h}_0) \neq \gamma(\mathbf{Y}_0).$$

We set $\mathbf{X}_0 = \Phi^{-1}(\mathbf{Y}_0)$.

On the one hand, we have

$$\mu(\mathbf{X}_0) = \gamma(\Phi(\mathbf{X}_0)) = \gamma(\Phi(\Phi^{-1}(\mathbf{Y}_0))) = \gamma(\mathbf{Y}_0).$$

On the other hand, we have

$$\begin{aligned} \mu(\Phi \circ \Phi_{\mathbf{h}_0} \circ \Phi^{-1}(\mathbf{X}_0)) &= \gamma(\Phi \circ \Phi_{\mathbf{h}_0} \circ \Phi^{-1} \circ \Phi(\mathbf{X}_0)) \\ &= \gamma(\Phi(\mathbf{X}_0) + \mathbf{h}_0) = \gamma(\mathbf{Y}_0 + \mathbf{h}_0). \end{aligned}$$

Thus, we deduce that $\Phi \circ \Phi_{\mathbf{h}_0} \circ \Phi^{-1} \in \mathbb{Q}_\mu \setminus \mathbb{I}_\mu$.

This ends the proof of Theorem 2.1. \square

Theorem 2.2. *Let μ be a diffused probability Borel measure defined on the Polish space \mathbb{E} . Then we have*

$$\mathbf{card}(\mathbb{Q}_\mu) = \mathbf{card}(\mathbb{I}_\mu) = \mathbf{c},$$

where \mathbf{c} denotes the cardinality of the continuum.

Proof. Let us denote by $\mathbb{A}(\mathbb{E})$, the group of all Borel automorphisms of \mathbb{E} . Since $\mathbb{I}_\mu \subseteq \mathbb{Q}_\mu \subseteq \mathbb{A}(\mathbb{E})$ and $\mathbf{card}(\mathbb{A}(\mathbb{E})) \leq \mathbf{c}$, we have

$$\mathbf{card}(\mathbb{I}_\mu) \leq \mathbf{card}(\mathbb{Q}_\mu) \leq \mathbf{card}(\mathbb{A}(\mathbb{E})) \leq \mathbf{c}.$$

Let \mathbf{S} be the unit circle in the Euclidean plane \mathbb{R}^2 . We may identify the circle \mathbf{S} with a compact group of all rotations of \mathbb{R}^2 about its origin. Let λ be the probability Haar measure defined on the compact group \mathbf{S} . By Lemma 2.1, there is a Borel isomorphism $\Phi : \mathbb{E} \rightarrow \mathbf{S}$ such that

$$(\forall \mathbf{X})(\mathbf{X} \in \mathcal{B}(\mathbb{E}) \rightarrow \mu(\mathbf{X}) = \lambda(\Phi(\mathbf{X}))).$$

Let us consider a group \mathbf{G}_0 of measurable automorphisms of \mathbb{E} defined by

$$\mathbf{G}_0 = \{\Phi \circ \mathbf{T}_g \circ \Phi^{-1} : g \in \mathbf{S}\},$$

where $\mathbf{T}_g(\mathbf{h}) = g\mathbf{h}$ for $g, \mathbf{h} \in \mathbf{S}$.

It is clear that $\mathbf{card}(\mathbf{G}_0) = \mathbf{c}$. Now, let us show that $\mathbf{G}_0 \subseteq \mathbb{I}_\mu$. Indeed, for $\Phi \circ \mathbf{T}_g \circ \Phi^{-1} \in \mathbf{G}_0$, we have

$$\begin{aligned} (\forall X)(X \in \mathcal{B}(\mathbb{E})) &\rightarrow \mu(\Phi \circ \mathbf{T}_g \circ \Phi^{-1}(X)) \\ &= \lambda(\Phi(\Phi \circ \mathbf{T}_g \circ \Phi^{-1}(X))) \\ &= \lambda(\Phi(\Phi^{-1}(\mathbf{T}_g(\Phi(X)))))) \\ &= \lambda(\mathbf{T}_g(\Phi(X))) = \lambda(\Phi(X)) = \mu(X). \end{aligned}$$

The latter relations mean that

$$\mathbf{card}(\mathbb{I}_\mu) \geq \mathbf{c}.$$

This ends the proof of Theorem 2.2. □

Theorem 2.3. *Let μ be a diffused probability Borel measure defined on the Radon metric space \mathbb{E} . Then we have*

$$\mathbb{A}(\mathbb{E}) \setminus \mathbb{Q}_\mu \neq \emptyset.$$

Proof. By Lemma 2.2, there exists a closed separable subspace $\mathbb{E}(\mu)$ of the \mathbb{E} such that

$$\mu(\mathbb{E} \setminus \mathbb{E}(\mu)) = 0.$$

Note that the closed separable subset $\mathbb{E}(\mu)$ of the complete metric space (\mathbb{E}, ρ) with the restriction ρ_1 of the metric ρ to the set $\mathbb{E}(\mu) \times \mathbb{E}(\mu)$ is a Polish space. Let γ be a standard Gaussian measure on \mathbb{R}^∞ .

By Lemma 2.1, there is a Borel isomorphism

$$\Phi : \mathbb{E}(\mu) \rightarrow \mathbb{R}^\infty,$$

such that $(\forall \mathbf{X})(\mathbf{X} \in \mathcal{B}(\mathbb{E}(\mu)) \rightarrow \mu(\mathbf{X}) = \gamma(\Phi(\mathbf{X})))$.

Let us consider a group \mathbf{G}_0 of Borel automorphisms of the $\mathbb{E}(\mu)$ defined by

$$\mathbf{G}_0 = \{\Phi \circ \Phi_{\mathbf{h}} \circ \Phi^{-1} : \mathbf{h} \in \mathbb{R}^\infty\},$$

where $\Phi_{\mathbf{h}}$ is a shift in \mathbb{R}^∞ defined by $(\forall \mathbf{x})(\mathbf{x} \in \mathbb{R}^\infty \rightarrow \Phi_{\mathbf{h}}(\mathbf{x}) = \mathbf{x} + \mathbf{h})$.

For $h \in \mathbb{R}^\infty$, we set $g_h(x) = \Phi \circ \Phi_h \circ \Phi^{-1}(x)$, for $x \in \mathbb{E}(\mu)$ and $g_h(x) = x$, for $x \in \mathbb{E} \setminus \mathbb{E}(\mu)$.

Following Kakutani [16], we have

$$(\exists \mathbf{h}_0)(\exists \mathbf{X}_0)(\mathbf{h} \in \mathbb{R}^\infty \setminus \ell_2 \ \& \ \mathbf{X}_0 \in \mathcal{B}(\mathbb{R}^\mathbb{N}) \rightarrow \gamma(\mathbf{X}_0) = \mathbf{0} \ \& \ \gamma(\mathbf{X}_0 + \mathbf{h}_0) > \mathbf{0}).$$

Now, it is not difficult to show that

$$\Phi \circ \Phi_{\mathbf{h}_0} \circ \Phi^{-1} \in \mathbb{A}(\mathbb{E}) \setminus \mathbb{Q}_\mu.$$

This ends the proof of Theorem 2.3. □

3. On \mathbb{T} -cm and \mathbb{T} -shy Sets in Complete Metric Groups

Let \mathbf{G} be a complete metric group, by which we mean a group with a complete metric for which the transformation (from $\mathbf{G} \times \mathbf{G}$ onto \mathbf{G}), which sends (\mathbf{x}, \mathbf{y}) into $\mathbf{x}^{-1}\mathbf{y}$ is continuous. Let $\mathcal{B}(\mathbf{G})$ denotes the σ -algebra of Borel subsets of \mathbf{G} .

Note that if \mathbf{G} is a complete metric group, then $\text{card}(\mathbf{G}) \leq c$ (see, [43], Proposition 2.2, Item(b), p. 24). Indeed, if we assume the contrary and $\text{card}(\mathbf{G}) > c$, then the function $f : \mathbf{G} \times \mathbf{G} \rightarrow \mathbf{G}$ defined by $f(x, y) = x^{-1}y$, for $(x, y) \in \mathbf{G} \times \mathbf{G}$ is not measurable, because

$$f^{-1}(\mathbf{0}) = \{(x, x) : x \in \mathbf{G}\} \notin \mathcal{B}(\mathbf{G}),$$

where $\mathbf{0}$ denotes a unit of the group \mathbf{G} . Moreover, the f is not continuous, and we get a contradiction with the definition of the complete metric group \mathbf{G} .

For $\mathbf{g} \in \mathbf{G}$, we define right $\mathbf{T}_{\mathbf{g}}$ and left ${}_{\mathbf{g}}\mathbf{T}$ shifts of the \mathbf{G} by

$$(\forall \mathbf{h})(\mathbf{h} \in \mathbf{G} \rightarrow \mathbf{T}_{\mathbf{g}}(\mathbf{h}) = \mathbf{hg} \ \& \ {}_{\mathbf{g}}\mathbf{T}(\mathbf{h}) = \mathbf{gh}).$$

Let \mathbb{T} be a subgroup of the $\mathbb{A}(\mathbf{G})$ consisting all left and right shifts of the \mathbf{G} .

Definition 3.1. A Borel set $\mathbf{X} \subseteq \mathbf{G}$ will be called \mathbb{T} -cm set (in the sense of Christensen and Mycielski), if there exists a Borel probability measure \mathbf{m} over \mathbf{G} , such that $\mathbf{m}(\mathbf{f}(\mathbf{X})) = \mathbf{0}$ for all $\mathbf{f} \in \mathbb{T}$. Every subset \mathbf{X}' of the \mathbf{X} will be called also \mathbb{T} -cm set.

A measure \mathbf{m} will be called a \mathbb{T} -testing measure to \mathbf{X} (or \mathbf{X}').

Definition 3.2. A Borel set $\mathbf{X} \subseteq \mathbf{G}$ will be called \mathbb{T} -shy (in the sense of Hunt et al. [13]), if

(i) there exists a Borel measure \mathbf{m} over \mathbf{G} such that $\mathbf{m}(f(\mathbf{X})) = \mathbf{0}$, for all $f \in \mathbb{T}$;

(ii) there exists a compact set $U \subseteq \mathbf{G}$, for which $0 < \mathbf{m}(U) < +\infty$.

Every subset \mathbf{X}' of the \mathbf{X} will be called also \mathbb{T} -shy set.

A measure \mathbf{m} will be called \mathbb{T} -transversal to \mathbf{X} (or \mathbf{X}').

Theorem 3.1. *Let \mathbf{G} be a Radon metric group. Let \mathbb{T} be a subgroup of the $\mathbb{A}(\mathbf{G})$ consisting all left and right shifts of the \mathbf{G} . Then for $X \subset \mathbf{G}$, we have that X is \mathbb{T} -cm set iff X is \mathbb{T} -shy set.*

Proof. Necessity. Let X is \mathbb{T} -cm set and let m be a \mathbb{T} -testing measure to the X , i.e., $\mathbf{m}(f(\mathbf{X})) = \mathbf{0}$, for all $f \in \mathbb{T}$. Since G is a Radon metric group, we claim that there exists a compact set $U \subseteq G$ for which $0 < \mathbf{m}(U) < +\infty$. Thus, \mathbf{m} is \mathbb{T} -transversal to \mathbf{X} and X is \mathbb{T} -shy set.

Sufficiency. Let X be a \mathbb{T} -shy set in \mathbf{G} and let μ be \mathbb{T} -transversal to \mathbf{X} . Then, we have

(i) $\mu(f(X)) = 0$, for all $f \in \mathbb{T}$;

(ii) there exists a compact set $U \subseteq \mathbf{G}$ for which $0 < \mu(U) < +\infty$.

We set $\mathbf{m}(Y) = \frac{\mu(U \cap Y)}{\mu(U)}$ for $Y \in \mathcal{B}(\mathbf{G})$. It is clear that m is a Borel probability measure. Let us show that \mathbf{m} is a \mathbb{T} -testing measure to X . Indeed, for $f \in \mathbb{T}$, we have

$$\mathbf{m}(f(X)) = \frac{\mu(U \cap f(X))}{\mu(U)} \leq \frac{\mu(f(X))}{\mu(U)} = 0.$$

Thus, X is \mathbb{T} -cm set. □

Definition 3.3. A set $\mathbf{X} \subseteq \mathbf{G}$ will be called \mathbb{T} -prevalent iff the complement of \mathbf{X} is \mathbb{T} -shy.

The class of all \mathbb{T} -shy sets we denote by $\mathbb{TS}(\mathbf{G})$.

Remark 3.1. Let \mathbb{T}_1 and \mathbb{T}_2 be subgroups of the $\mathbb{A}(\mathbf{G})$ consisting all left and right shifts of the \mathbf{G} such that $\mathbb{T}_1 \subseteq \mathbb{T}_2$. Then every \mathbb{T}_2 -shy set is at the same time \mathbb{T}_1 -shy set. The converse relation, in general, is not valid. Indeed, let \mathbf{G} be an abelian Polish group of all real numbers \mathbb{R} , \mathbb{T}_1 be a group of all shifts of the \mathbb{R} , and $\mathbb{T}_2 = \mathbb{A}(\mathbb{R})$. Let μ be a standard Gaussian probability measure in \mathbb{R} . By Theorem 2.3, we claim that there exists an element $g \in \mathbb{A}(\mathbb{R}) \setminus \mathbb{Q}_\mu$ and a \mathbb{T}_1 -shy set \mathbf{X}_0 such that $\mu(g(\mathbf{X}_0)) > 0$. Now, let ν be \mathbb{T}_2 -transversal to \mathbf{X}_0 . Then ν will be also \mathbb{T}_2 -transversal to $g(\mathbf{X}_0)$. It is clear that, then ν will be \mathbb{T}_1 -transversal to $g(\mathbf{X}_0)$, which implies that $g(\mathbf{X}_0)$ is \mathbb{T}_1 -shy set, equivalently, Gaussian null set and we obtain a contradiction.

Remark 3.2. Definitions 3.1 and 3.2 are generalizations of notions of null sets introduced in [6], [13], [20], respectively. Indeed,

(i) if \mathbf{G} is an abelian Polish group and \mathbb{T} is a group of all shifts of the \mathbf{G} , then a notion of \mathbb{T} -shy sets coincides with the notion of Christensen's Haar null sets introduced in [6].

(ii) if \mathbf{G} is a non-abelian Polish group and \mathbb{T} is a subgroup of the $\mathbb{A}(\mathbf{G})$ generated by left and right shifts of the \mathbf{G} , then a notion of \mathbb{T} -shy (equivalently, \mathbb{T} -cm) sets coincides with the notion of Mycielski's shy sets introduced in [20].

(iii) if \mathbf{G} is a topological vector space with complete metric and \mathbf{T} is a group of all shifts of the \mathbf{G} , then a notion of \mathbb{T} -shy (equivalently, \mathbb{T} -cm) sets coincides with the notion of shy set introduced by Hunt et al. in [13].

Fact 3.1. Let \mathbf{G} be a complete metric group. If S is \mathbb{T} -shy, then so is every subset of S and every $g(S)$ for $g \in \mathbb{T}$.

Proof. Let m be \mathbb{T} -transversal to S , i.e., $m(f(S)) = 0$, for all $f \in \mathbb{T}$.

For $S' \subseteq S$, we get $\overline{m}(f(S')) \leq m(f(S)) = 0$.

For $g, f \in \mathbb{T}$, we get $m(f(g(S))) = m((g \circ f)(S)) = 0$ because S is \mathbb{T} -shy and $g \circ f \in \mathbb{T}$. \square

Fact 3.2. Let G be a complete metric group. Every \mathbb{T} -shy Borel set S has a \mathbb{T} -transversal measure, which is finite with compact support. Furthermore, the support of this measure can be taken to have arbitrarily small diameter.

Proof. Let m be a \mathbb{T} -transversal measure to a Borel set $S \subset G$. Following Definition 3.2, there exists a compact set U of finite and positive measure. Then, it can be restricted to a compact set U , and the restriction m_U is certainly also a \mathbb{T} -transversal measure to S . Indeed, if U is of finite and positive measure, then

$$m_U(f(S)) = m(U \cap f(S)) \leq m(f(S)) = 0.$$

Also, since U is compact, it can be covered for each $\epsilon > 0$ by a finite number of balls of radius ϵ , and at least, one of these balls must intersect U in a set of positive measure. The intersection of U with the closure of this ball is compact, and the restriction of m to this set is also a \mathbb{T} -transversal measure to S . \square

In the proof of the next assertion, we repeat the scheme of Mycielski [20] with small, but important alternations.

Theorem 3.2. *Let G be a complete metric group. Then, the union of a countable collection of \mathbb{T} -shy sets is \mathbb{T} -shy.*

Proof. Given a countable collection of \mathbb{T} -shy subsets of G , let $(Y_k)_{k \in \mathbb{N}}$ be \mathbb{T} -shy Borel sets containing the original sets. For $n = 1, 2, \dots$, let m_n be a \mathbb{T} -transversal to Y_n . By Fact 3.2, we can assume without loss of generality that, each m_n is finite and supported on a compact set U_n with diameter at most 2^{-n} . Since \mathbb{T} contains all left and right shifts, by normalizing and translating the measures, we can also assume that $m_n(U_n) = 1$ for all n and that the unity of G belongs to U_n . Notice that, if $y_n \in U_n$ for $n = 1, 2, \dots$, then the infinite product $g_1 g_2 \dots$ in the sense of group multiplication converges (by the assumptions about diameters of the U_n 's).

The infinite Cartesian product $U^\Pi = U_1 \times U_2 \times \dots$ is compact by the well known Tychonoff theorem and has a product measure $m^\Pi = m_1 \times m_2 \times \dots$ defined on its Borel subsets, with $m^\Pi(U^\Pi) = 1$. Since G is complete and each vector in U_n lies at most 2^{-n} away from zero, there is a continuous mapping from U^Π into G defined by

$$(g_1, g_2, \dots) \mapsto g_1 g_2 \dots.$$

The image U of U^Π under this mapping is compact, and m^Π induces a measure m supported on U , given by

$$m(X) = m^\Pi(\{(g_1, g_2, \dots) \in U^\Pi : g_1 g_2 \dots \in X\}).$$

We will be done, if we show that m is a \mathbb{T} -transversal to Y_i for $i = 1, 2, \dots$. For $f \in \mathbb{T}$, we have

$$g_1 g_2 \dots \in f(Y_n) \Leftrightarrow g_n \in (g_1 \dots g_{n-1})^{-1} f(Y_n) (g_{n+1} g_{n+2} \dots)^{-1}.$$

Since the Cartesian product of measures is associative (and commutative), we can write

$$m^\Pi = m_n \times \nu_n^\Pi,$$

with

$$\nu_n^\Pi = m_1 \times \dots \times m_{n-1} \times m_{n+1} \times \dots.$$

Let $\prod^n U_n = U_1 \times \dots \times U_{n-1} \times U_{n+1} \times U_{n+2} \times \dots$. Then, since m_n is \mathbb{T} -transversal to Y_n and $f \circ_{(g_1 \dots g_{n-1})^{-1}} T \circ_{(g_{n+1} g_{n+2} \dots)^{-1}} \in \mathbb{T}$, we claim

$$m(f(Y_n)) = \int_{\prod^n U_n} m_n((g_1 \dots g_{n-1})^{-1} f(Y_n) (g_{n+1} g_{n+2} \dots)^{-1}) \nu_n^\Pi(dg) = 0.$$

So, m is a \mathbb{T} -transversal to each Y_n and hence also to $Y_1 \cup Y_2 \cup \dots$. \square

Remark 3.3. The main result of [13] can be obtained from Theorem 3.2, when \mathbf{G} is a complete metric topological vector space and the \mathbb{T} coincides with the group of all left and right shifts of the \mathbf{G} .

We get the following corollary of Theorem 3.2.

Corollary 3.1 (CH). *Let \mathbf{G} be a complete metric group. Then, the collection of \mathbb{T} -cm sets and \mathbb{T} -shy sets, constituting σ -ideals, coincide.*

Remark 3.4. On the one hand, under (CH) Corollary 3.1 covers simultaneously both main results obtained in [13] and [20], respectively, because the class of Radon metric groups is more large than the class of Polish groups (see Example 3.1).

By the scheme used in the proof of Theorem 3.2, we get:

Theorem 3.3. *Let \mathbf{G} be a Radon metric group. Then, the collection of \mathbb{T} -cm sets constitute a σ -ideal.*

The following example demonstrates that the class of Radon metric groups is more wider than the class of Polish groups.

Example 3.1. Let I be a set of cardinality \aleph_1 . Let denote by $(L_2(I), \rho)$, a complete metric space of all real-valued square-summable sequences on I , where a metric ρ is defined by

$$\rho((x_i)_{i \in I}, (y_i)_{i \in I}) = \left(\sum_{i \in I} (x_i - y_i)^2 \right)^{\frac{1}{2}},$$

for $(x_i)_{i \in I}, (y_i)_{i \in I} \in L_2(I)$.

It is clear that this is an invariant metric on $L_2(I)$, which produced a norm $\|\cdot\|_{L_2(I)}$ as follows

$$\|(x_i)_{i \in I}\|_{L_2(I)} = \rho((x_i)_{i \in I}, \mathbf{0}),$$

where $\mathbf{0}$ denotes the zero of the vector space $L_2(I)$.

Let us show that $(L_2(I), \rho)$ is a metric group. Indeed, if $((x_i^{(n)})_{i \in I})_{n \in \mathbb{N}}$ and $((y_i^{(n)})_{i \in I})_{n \in \mathbb{N}}$ tend to $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$, respectively, then we have

$$\begin{aligned}
& \rho((y_i)_{i \in I} - (x_i)_{i \in I}, (y_i^{(n)})_{i \in I} - (x_i^{(n)})_{i \in I}) \\
&= \rho(((y_i)_{i \in I} - (x_i)_{i \in I}) - ((y_i^{(n)})_{i \in I} - (x_i^{(n)})_{i \in I}), \mathbf{0}) \\
&= \|((y_i)_{i \in I} - (y_i^{(n)})_{i \in I}) - ((x_i)_{i \in I} - (x_i^{(n)})_{i \in I})\|_{L_2(I)} \\
&\leq \|(y_i)_{i \in I} - (y_i^{(n)})_{i \in I}\|_{L_2(I)} + \|(x_i)_{i \in I} - (x_i^{(n)})_{i \in I}\|_{L_2(I)} \rightarrow 0,
\end{aligned}$$

when n tends to $+\infty$.

It is obvious that a topological weight of $(L_2(I), \rho)$ is equal to \aleph_1 . By famous result of Ulam, we claim that the topological weight of $(L_2(I), \rho)$ being equal to the \aleph_1 is not a real-valued measurable cardinal. By Lemma 2.2, we deduce that $L_2(I)$ is Radon metric abelian group. Now, let \mathbf{G}_1 be a non-abelian Polish group. We put $\mathbf{G} = \mathbf{G}_1 \times L_2(I)$. Then, it is obvious to see that \mathbf{G} is a Radon non-Polish non-abelian metric group.

Remark 3.5. Since every Polish group is Radon metric group and the converse relation is not valid (see, Example 3.1), we claim that Theorem 3.3 covers the main result of [20] (see, Theorem 3, p. 32). In particular, Mycielski's famous result can be obtained from Theorem 3.3, when \mathbf{G} is a Polish group and the \mathbb{T} coincides with the group of all left and right shifts of the \mathbf{G} .

4. On Generators of (\mathbb{T}, μ) - shy Sets in Complete Metric Groups

Let \mathbf{G} be a complete metric group.

Definition 4.1. A Borel measure μ in \mathbf{G} is called a *generator* of \mathbb{T} -shy sets in \mathbf{G} , if

$$(\forall \mathbf{X})(\bar{\mu}(\mathbf{X}) = \mathbf{0} \rightarrow \mathbf{X} \in \mathbb{TS}(\mathbf{G})),$$

where $\bar{\mu}$ denotes a usual completion of the Borel measure μ .

Definition 4.2. A Borel measure μ in \mathbf{G} is called *quasi-finite*, if there exists a compact set $U \subseteq \mathbf{G}$ for which $0 < \mu(U) < \infty$.

Definition 4.3. A Borel measure μ in \mathbf{G} is called *semi-finite*, if for X with $\mu(X) > 0$, there exists a compact subset $F \subseteq X$ for which $0 < \mu(F) < \infty$.

Definition 4.4. A Borel measure μ in \mathbf{G} is called \mathbb{T} -*quasi-invariant*, if μ and μ^f , defined by

$$(\forall X)(X \in \mathcal{B}(\mathbf{G}) \rightarrow \mu^f(X) = \mu(f(X))),$$

are equivalent for $f \in \mathbb{T}$.

Definition 4.5. A Borel measure μ in \mathbf{G} is called \mathbb{T} -*invariant*, if

$$(\forall X)(\forall f)(X \in \mathcal{B}(\mathbf{G}) \ \& \ f \in \mathbb{T} \rightarrow \mu(f(X)) = \mu(X)).$$

Definition 4.6. A set $X \subset \mathbf{G}$ is called (\mathbb{T}, μ) -*shy*, if it is a subset of a Borel set S' with $\mu(f(S')) = 0$, for $f \in \mathbb{T}$. The class of all (\mathbb{T}, μ) -shy sets is denoted by \mathbb{TS}_μ .

It is clear that

$$\mathbb{TS}(\mathbf{G}) = \bigcup_\mu \mathbb{TS}_\mu.$$

Definition 4.7. A Borel measure λ in a complete metric group \mathbf{G} is called a *generator* of (\mathbb{T}, μ) -shy sets, if the following equality

$$\mathbb{TS}_\mu = N(\bar{\lambda})$$

holds, where $N(\bar{\lambda})$ denotes the class of $\bar{\lambda}$ measure zero.

Definition 4.8. Let μ be a Borel measure in a complete metric group \mathbf{G} . A set

$$\{f : f \in \mathbb{T} \ \& \ \mu^f \sim \mu\}$$

is denoted by $\mathbb{Q}_{(\mathbb{T}, \mu)}$.

It is easy to show that $\mathbb{Q}_{(\mathbb{T}, \mu)}$ is a subgroup of the group \mathbb{T} . Moreover, the following equalities

$$\mathbb{Q}_{(\mathbb{T}, \mu)} = \mathbb{Q}_\mu \cap \mathbb{T},$$

and

$$\mathbb{Q}_{(\mathbb{A}(\mathbf{G}), \mu)} = \mathbb{Q}_\mu,$$

hold.

For $f_1, f_2 \in \mathbb{T}$, we set $f_1 \sim f_2$ iff $f_2^{-1} \circ f_1 \in \mathbb{Q}_{(\mathbb{T}, \mu)}$. It is clear the binary relation \sim on the \mathbb{T} is an equivalence relation since it is reflexive, symmetric, and transitive. Let $(K_i)_{i \in I}$ be the partition of the \mathbb{T} (this is unique) defined by \sim on \mathbb{T} . Let i_0 be such an index that $K_{i_0} = \mathbb{Q}_{(\mathbb{T}, \mu)}$.

Let τ be an operator of global choice. We set

$$\mathbb{Q}_{(\mathbb{T}, \mu)}^\perp = \{\tau(K_i) : i \in I \setminus \{i_0\}\} \cup \{e\},$$

where e denotes an identity transformation of the group \mathbf{G} .

We do not know whether $\mathbb{Q}_{(\mathbb{T}, \mu)}^\perp$ is a subgroup of the group \mathbb{T} , but the following equality $\mathbb{Q}_{(\mathbb{T}, \mu)}^\perp \circ \mathbb{Q}_{(\mathbb{T}, \mu)} = \mathbb{T}$ always holds.

It is clear that $\mathbb{Q}_{(\mathbb{T}, \mu)} = \mathbb{T}$, if and only if the μ is \mathbb{T} -quasi-invariant.

Definition 4.9. Let μ be a Borel measure in \mathbf{G} . A set

$$\{f : f \in \mathbb{T} \ \& \ \mu^f = \mu\}$$

is denoted by $\mathbb{I}_{(\mathbb{T}, \mu)}$.

It can be shown that, $\mathbb{I}_{(\mathbb{T}, \mu)} = \mathbb{T}$, if and only if the μ is \mathbb{T} -translation-invariant. Note that the following two conditions are fulfilled:

- (i) $\mathbb{I}_{(\mathbb{T}, \mu)} = \mathbb{T} \cap \mathbb{I}_\mu$,
- (ii) $\mathbb{I}_\mu = \mathbb{I}_{(\mathbb{A}(\mathbf{G}), \mu)}$.

Let μ be an arbitrary Borel measure in \mathbf{G} .

Let us define a functional $\mathbf{G}_{(\mathbb{T}, \mu)}$ by

$$(\forall X)(X \in \mathcal{B}(\mathbf{G}) \rightarrow \mathbf{G}_{(\mathbb{T}, \mu)}(X) = \sum_{f \in \mathbb{Q}_{(\mathbb{T}, \mu)}^\perp} \mu(f(X))).$$

Note here that, if \mathbf{G}_0 is any selector of the above-mentioned partition $(K_i)_{i \in I}$ of the \mathbb{T} , then the following condition holds

$$(\forall X)(X \in \mathcal{B}(\mathbf{G}) \rightarrow (\mathbf{G}_{(\mathbb{T}, \mu)}(X) = 0 \Leftrightarrow \sum_{f \in \mathbf{G}_0} \mu(f(X)) = 0)).$$

Definition 4.10. A quasi-finite Borel measure μ in a complete metric group \mathbf{G} is called a \mathbb{T} -quasi-generator, if there exists a σ -compact F such that

$$\begin{aligned} & (\forall f_1, f_2)(f_1 \in \mathbb{Q}_{(\mathbb{T}, \mu)} \ \& \ f_2 \in \mathbb{Q}_{(\mathbb{T}, \mu)}^\perp \setminus \{e\} \rightarrow \\ & \mu(\mathbf{G} \setminus (F \cap (f_1(F)))) = 0 \ \& \ \mu(F \cap (f_2(F))) = 0). \end{aligned}$$

Theorem 4.1. Let μ be an arbitrary Borel measure in a complete metric group \mathbf{G} . Then,

(i) $\mathbf{G}_{(\mathbb{T}, \mu)}$ is a generator of \mathbb{T} -shy sets in \mathbf{G} such that

$$\mathbb{T}\mathbb{S}_\mu = N(\overline{\mathbf{G}_{(\mathbb{T}, \mu)}}),$$

where $\overline{\mathbf{G}_{(\mathbb{T}, \mu)}}$ denotes a usual completion of the $\mathbf{G}_{(\mathbb{T}, \mu)}$.

(ii) If μ is a \mathbb{T} -quasi-generator, then the generator $\mathbf{G}_{(\mathbb{T}, \mu)}$ is quasi-finite.

Proof. (i) We have to show that $\mathbb{T}\mathbb{S}_\mu = N(\overline{\mathbf{G}_{(\mathbb{T}, \mu)}})$.

Let $X \in N(\overline{\mathbf{G}_{(\mathbb{T}, \mu)}})$. This means that, there exists a Borel set X' for which $X \subseteq X'$ and $\mathbf{G}_{(\mathbb{T}, \mu)}(X') = 0$.

For $f \in \mathbb{T}$, we have the representation $f = f_2 \circ f_1$, where $f_1 \in \mathcal{Q}_{(\mathbb{T}, \mu)}$ and $f_2 \in \mathcal{Q}_{(\mathbb{T}, \mu)}^\perp$.

The relation $0 = \mathbf{G}_{(\mathbb{T}, \mu)}(X') = \sum_{u \in \mathcal{Q}_{(\mathbb{T}, \mu)}^\perp} \mu(u(X'))$ implies that $\mu(f_2(X')) = 0$, since $f_2 \in \mathcal{Q}_{(\mathbb{T}, \mu)}^\perp$. The condition $f_1 \in \mathcal{Q}_{(\mathbb{T}, \mu)}$ implies that $\mu(f_1(f_2(X'))) = 0$. By the representation $f = f_2 \circ f_1$, we claim that $\mu(f(X')) = 0$.

Since $f \in \mathbb{T}$ was taken arbitrarily, we claim that $X' \in \mathbb{TS}_\mu$.

Let $X \in \mathbb{TS}_\mu$. This means that, there exists a Borel set X' such that $X \subseteq X'$ and $\mu(u(X')) = 0$ for $u \in \mathbb{T}$. The latter relation means that

$$\sum_{u \in \mathcal{Q}_{(\mathbb{T}, \mu)}^\perp} \mu(u(X')) = 0,$$

which implies that $X' \in \mathcal{N}(\mathbb{G}_{(\mathbb{T}, \mu)})$ and $X \in \mathcal{N}(\overline{\mathbf{G}}_{(\mathbb{T}, \mu)})$ because $X \subseteq X'$.

(ii) Let μ be a \mathbb{T} -quasi-generator, i.e., μ is a quasi-finite and there exists a σ -compact F such that

$$\begin{aligned} & (\forall f_1, f_2)(f_1 \in \mathcal{Q}_{(\mathbb{T}, \mu)} \ \& \ f_2 \in \mathcal{Q}_{(\mathbb{T}, \mu)}^\perp \setminus \{e\} \rightarrow \\ & \mu(\mathbf{G} \setminus (F \cap (f_1(F)))) = 0 \ \& \ \mu(F \cap (f_2(F))) = 0). \end{aligned}$$

Since $\mu(\mathbf{G} \setminus F) = 0$ and μ is quasi-finite, there exists a compact set $U \subset F$ with $0 < \mu(U) < \infty$. Thus, we have

$$(\forall u)(u \in \mathcal{Q}_{(\mathbb{T}, \mu)}^\perp \setminus \{e\} \rightarrow \mu(u(U)) = 0),$$

because

$$\mu(u(U)) = \mu(u(U) \cap F) \leq \mu(u(F) \cap F) = 0,$$

for $u \in \mathcal{Q}_{(\mathbb{T}, \mu)}^\perp \setminus \{e\}$.

Hence

$$\mathbb{G}_{(\mathbb{T}, \mu)}(U) = \sum_{u \in \mathbb{Q}_{(\mathbb{T}, \mu)}^\perp} \mu(u(U)) = \mu(e(U)) = \mu(U).$$

Thus, $\mathbb{G}_{(\mathbb{T}, \mu)}$ is a quasi-finite generator of (\mathbb{T}, μ) -shy sets in \mathbf{G} . \square

5. On Quasi-finiteness of the Generator of $(\pi(\mathbb{V}), \mu)$ -shy Sets in a Polish Topological Vector Space V

Let \mathbb{V} be a Polish topological vector space and $\pi(\mathbb{V})$ be a group of all shifts of the \mathbb{V} . In [30] has been established the validity of the partial case of Theorem 3.2.

Theorem 5.1 ([30], Theorem 3.1, p. 245). *Let μ be an arbitrary Borel measure in \mathbb{V} . Then $\mathbf{G}_{(\pi(\mathbb{V}), \mu)}$ is a generator of $(\pi(\mathbb{V}), \mu)$ -shy sets in \mathbb{V} , such that*

$$\pi(\mathbb{V})\mathbb{S}_\mu = N(\overline{\mathbf{G}}_{(\pi(\mathbb{V}), \mu)}).$$

If μ is a $\pi(\mathbb{V})$ -quasi-generator, then the generator $\mathbf{G}_{(\pi(\mathbb{V}), \mu)}$ is quasi-finite.

In the context of Theorem 5.1, the following problem has been posted in [30].

Problem 5.1. For a diffused Borel measure μ in a Polish topological vector space V , whether there exists a quasi-finite generator of $(\pi(V), \mu)$ -shy sets in V ?

Note that Problem 5.1 is not trivial and its solution depends on the structure of the measure μ . In [30], Problem 5.1 has been solved for $\pi(\mathbb{V})$ -quasi-generators. In particular, it has been proved that Kharazisvili's measure [18] defined in the $\mathbb{R}^{\mathbb{N}}$ is $\pi(\mathbb{R}^{\mathbb{N}})$ -quasi-generator.

A certain purpose of the present section is to solve Problem 5.1 for a Borel probability measure μ for which $\mu(Q_\mu) = 1$.

Remark 5.1. Let E be a separable Hilbert space and μ be a Borel probability measure on E . Following Skorohod [39] (see 19), the following conditions hold:

(S1) $Q_{(\pi(E), \mu)}$ is a Borel subset of E ;

(S2) if $\mu(F) = 0$ for every finite-dimensional subspace $F \subset E$, then $\mu(Q_{(\pi(E), \mu)}) = 0$;

(S3) if $\mu(Q_{(\pi(E), \mu)}) = 1$, then there exists a sequence $(L_i)_{i \in \mathbb{N}}$ of finite-dimensional subspaces such that $\mu(\bigcup_{i \in \mathbb{N}} L_i) = 0$.

On the other hand, Okazaki [21] obtained the following result:

Let G be a complete separable metrizable abelian topological group and μ be a probability measure on G . Then,

(O1) $Q_{(\pi(G), \mu)}$ is a Borel subset of G ;

(O2) if $\mu(Q_{(\pi(G), \mu)}) > 0$, then Q_μ is a locally compact σ -compact topological group with respect to the induced topology from G .

There are similarities between these two results. In fact, each locally compact locally convex Hausdorff space is finite-dimensional and each locally compact σ -compact subgroup of a locally convex Hausdorff space is of the form $R^n +$ (countable subgroup) (the structure theorem of the locally compact σ -compact abelian topological group).

In [23], the above results have been generalized as follows.

Theorem A. *Let E be a general locally convex Hausdorff space and $\mu^*(Q_{(\pi(E), \mu)}) > 0$, where μ^* denotes the outer measure. Then, there exist a finite-dimensional subspace L , a thick subgroup G of L , and a countable subgroup $\{x_i : i \in \mathbb{N}\}$ of E such that $Q_{(\pi(G), \mu)} = \bigcup_{i \in \mathbb{N}} (G + x_i)$.*

Theorem B. *Let E be a Souslin locally convex Hausdorff space. Then $Q_{(\pi(G), \mu)}$ is a Borel subset of E and if $\mu(Q_{(\pi(E), \mu)}) > 0$, $Q_{(\pi(E), \mu)}$ can be written as $Q_{(\pi(E), \mu)} = \bigcup_{i \in \mathbb{N}} (L + x_i)$, where L is a finite-dimensional subspace and $\{x_i : i \in \mathbb{N}\}$ is a countable subgroup of E .*

Remark 5.2. Let H be a subset of a locally convex Hausdorff space E and μ be a probability measure on $\mathcal{C}(E, E^*)$ the cylindrical σ -algebra generated by the topological dual E^* . We set $\tau_x(y) = y + x$, $\tau_x(x \in E)$ is $\mathcal{C}(E, E^*)$ -measurable. μ is said to be H -quasi-invariant, if holds $\tau_x(\mu) \sim \mu$ for every $x \in H$, where $\tau_x(\mu)(A) = \mu(A - x)$, $A \in \mathcal{C}(E, E^*)$. The H -quasi-invariant measure μ is said to be H -ergodic, if $\mu(A \Delta (A - x)) = 0$ for every $x \in H$, then $\mu(A) = 0$ or 1, where Δ denotes the symmetric difference.

In [22], has been obtained the following:

Theorem C. *Let E be a locally convex Hausdorff space, H_1 and H_2 be two linear subspaces of E , μ_1 and μ_2 be two probability measures on $\mathcal{C}(E, E^*)$, and assume that μ_i is H_i -quasi-invariant and H_i -ergodic ($i = 1, 2$). Then μ_1 and μ_2 are equivalent or singular.*

Theorem C unifies many known dichotomies such as (a) Kakutani's dichotomy for product measures on R^∞ (see [16]), (b) Hajek-Feldman's dichotomy [14], (c) Skorohod's dichotomy (see [39]), and so on.

In context of dichotomy of Gaussian measures, Gikhman and Skorohod considered the following problem in [10] (see Chapter 7, Paragraph 2).

Problem 5.2. Does there exist a probability Borel measure μ in the Hilbert space ℓ_2 , which satisfies the following conditions:

- (i) the group $Q_{(\pi(\ell_2), \mu)}$ is an everywhere dense linear manifold in ℓ_2 ;

(ii) there exists $a \in \ell_2 \setminus Q_{(\pi(\ell_2), \mu)}$ such that a measure μ is not orthogonal to the measure $\mu^{(a)}$, where

$$(\forall X)(X \in B(\ell_2) \rightarrow \mu^{(a)}(X) = \mu(X - a))?$$

Gikhman-Skorohod's positive solution of this problem employs the technique of Gaussian measures in an infinite-dimensional separable Hilbert space. In [26], Gikhman-Skorohod's result was extended to invariant Borel measures in ℓ_2 . In particular, a nonzero σ -finite Borel measure μ is constructed in ℓ_2 , which satisfies the following conditions:

(iii) the group $I_{(\pi(\ell_2), \mu)}$ is an everywhere dense linear manifold in ℓ_2 ;

(iv) there exists $a \in \ell_2 \setminus I_{(\pi(\ell_2), \mu)}$ such that a measure $\mu^{(a)}$ is not orthogonal to the measure μ .

Since every translation-quasi-invariant Borel measure in Polish topological vector spaces has a property that it and its every translate are equivalent, there naturally arises the following:

Problem 5.3. Let \mathcal{T} be a group of Borel measurable automorphisms of the Polish group G consisting all left and right shifts. Does there exist generators of \mathcal{T} -shy sets μ_1 and μ_2 in Polish group G , which are not orthogonal or equivalent?

In context of Problem 5.1, the following assertion is valid.

Theorem 5.2. *Let V be a Polish topological vector space and μ be a Borel probability measure in V . If $\mu(Q_{(\pi(V), \mu)}) = 1$, then the generator $G_{(\pi(V), \mu)}$ is quasi-finite and semi-finite.*

Proof. By Theorem B, $Q_{(\pi(V), \mu)}$ can be written as $Q_{(\pi(V), \mu)} = \bigcup_{i \in \mathbb{N}} (L + x_i)$, where L is a finite-dimensional subspace and $\{x_i : i \in \mathbb{N}\}$ is a countable subgroup of V . Since $\mu(Q_{(\pi(V), \mu)}) = 1$, we deduce that the measure μ is concentrated on the set $Q_{(\pi(V), \mu)} = \bigcup_{i \in \mathbb{N}} (L + x_i)$.

We have

$$(\forall X)(X \in \mathcal{B}(V) \rightarrow G_\mu(X) = \sum_{v \in \mathcal{Q}_{(\pi(V), \mu)}^\perp} \mu(X + v) = \sum_{v \in \mathcal{Q}_{(\pi(V), \mu)}^\perp} \mu((X + v) \cap \mathcal{Q}_{(\pi(V), \mu)})).$$

Using an inner regularity of the measure μ , we can choose such a compact set $U \subset \mathcal{Q}_{(\pi(V), \mu)}$ that $0 < \mu(U) < 1$. We get $G_{(\pi(V), \mu)}(U) = \mu(U)$. The latter relation means that $G_{(\pi(V), \mu)}$ is quasi-finite.

Now, we have to show that $G_{(\pi(V), \mu)}$ is semi-finite. Indeed, let for any $X \in \mathcal{B}(V)$, we have $G_{(\pi(V), \mu)}(X) > 0$. It means that there is $v_0 \in \mathcal{Q}_{(\pi(V), \mu)}^\perp$ such that $\mu(X + v_0 \cap \mathcal{Q}_{(\pi(V), \mu)}) > 0$. Using again an inner regularity of the measure μ , we can choose a compact set $U \subseteq X + v_0 \cap \mathcal{Q}_{(\pi(V), \mu)}$ such that

$$0 < \mu(U) < \mu(X + v_0 \cap \mathcal{Q}_{(\pi(V), \mu)}).$$

A condition $U - v_0 \subseteq X \cap (\mathcal{Q}_{(\pi(V), \mu)} - v_0)$ implies that $U - v_0 \subseteq X$. Hence, we have

$$\begin{aligned} G_{(\pi(V), \mu)}(U - v_0) &= \sum_{v \in \mathcal{Q}_{(\pi(V), \mu)}^\perp} \mu(U - v_0 + v \cap \mathcal{Q}_{(\pi(V), \mu)}) \\ &= \mu(U - v_0 + v_0 \cap \mathcal{Q}_{(\pi(V), \mu)}) + \sum_{v \in \mathcal{Q}_{(\pi(V), \mu)}^\perp \setminus \{v_0\}} \mu(U - v_0 + v \cap \mathcal{Q}_{(\pi(V), \mu)}) \\ &= \mu(U \cap \mathcal{Q}_{(\pi(V), \mu)}) + \sum_{v \in \mathcal{Q}_{(\pi(V), \mu)}^\perp \setminus \{v_0\}} \mu\{\emptyset\} = \mu(U \cap \mathcal{Q}_{(\pi(V), \mu)}). \end{aligned}$$

Since $U \subseteq \mathcal{Q}_{(\pi(V), \mu)}$ and $\mu(\mathcal{Q}_{(\pi(V), \mu)}) = 1$, we claim that $\mu(U \cap \mathcal{Q}_{(\pi(V), \mu)}) = \mu(U)$.

Finally, for a compact set $U - v_0$, we have $U - v_0 \subseteq X$ and $0 < G_{(\pi(V), \mu)}(U - v_0) < 1$. Thus, a semi-finiteness of the generator $G_{(\pi(V), \mu)}$ is proved. \square

Theorem 5.3. *Let V be a Polish topological vector space and μ be a Borel probability measure in V . If $\mu(Q_{(\pi(V), \mu)}) = 1$, then there exists a quasi-finite semi-finite translation-invariant generator of $(\pi(V), \mu)$ -shy sets $G_{(\pi(V), \nu)}$, which is equivalent to the generator $G_{(\pi(V), \mu)}$.*

Proof. By Theorem B, $Q_{(\pi(V), \mu)}$ can be written as $Q_{(\pi(V), \mu)} = \bigcup_{i \in \mathbb{N}} (L + x_i)$, where L is a finite-dimensional subspace and $\{x_i : i \in \mathbb{N}\}$ is a countable subgroup of V . Since $\mu(Q_{(\pi(V), \mu)}) = 1$, we deduce that a restriction μ_i of the μ to the measurable space $(L + x_i, \mathcal{B}(V) \cap (L + x_i))$ is an L -quasi-invariant finite non-zero Borel measure. The $Q_{(\pi(V), \mu)}$ -quasi-invariance of the measure μ implies that $\mu_i(L + x_i) > 0$, for $i \in \mathbb{N}$. Hence, for $i \in \mathbb{N}$, the μ_i is equivalent to the Lebesgue measure λ_i concentrated on $L + x_i$, and the measures ν and μ are equivalent, where ν is defined by

$$(\forall X)(X \in \mathcal{B}(V) \rightarrow \nu(X) = \sum_{i \in \mathbb{N}} \lambda_i(L + x_i \cap X)).$$

By this reason, the generators $G_{(\pi(V), \nu)}$ and $G_{(\pi(V), \mu)}$ are equivalent.

By the scheme used in the proof of Theorem 5.1, one can easily show that $G_{(\pi(V), \nu)}$ is a quasi-finite semi-finite translation-invariant generator of $(\pi(V), \nu)$ -shy sets in V . \square

In context of Theorem C, the following assertion is of some interest.

Theorem 5.4. *Let G be a Polish group and \mathcal{T} be a group of Borel automorphisms consisting all left and right shifts. Then, there does not exist a family of mutually singular generators of \mathcal{T} -shy sets in G .*

Proof. Assume the contrary and let $(\mu_i)_{i \in I}$ be any family of mutually singular generators of \mathcal{T} -shy sets. Let μ_{i_1} and μ_{i_0} be mutually singular generators of \mathcal{T} -shy sets from the family $(\mu_i)_{i \in I}$. Then, there

exists $A \in \mathcal{B}(G)$ such that $\mu_1(A) = 0$ and $\mu_2(B) = 0$, where $B = G \setminus A$. Since μ_{i_1} and μ_{i_0} are generators of \mathcal{T} -shy sets, we claim that both A and B are \mathcal{T} -shy sets in G . By Theorem 3.1, we claim that $A \cup B$, equivalently G , is \mathcal{T} -shy, which is a required contradiction and Theorem 5.4 is proved. \square

Remark 5.3. It is well known in folklore that, if (X, M, μ) is a σ -finite measure space, then for a σ -finite measure ν on M , there exist measures α and β such that $\alpha \ll \mu$ and $\beta \perp \mu$ and for which $\nu = \alpha + \beta$. Moreover, the measures α and β are unique. Such a decomposition is called “*Lebesgue decomposition*”, which always occurs for a σ -finite measure space. Following Theorem 4.5, we have a different picture for generators of \mathcal{T} -shy sets in Polish groups. More precisely, in a Polish group G , no any generator μ of \mathcal{T} -shy sets admits the following representation $\mu = \mu_1 + \mu_2$, where μ_1 and μ_2 are mutually singular generators of \mathcal{T} -shy sets in G .

Remark 5.4. As corollary of Theorem 4.5, we get that there do not exist orthogonal generators of $\pi(V)$ -shy sets in a Polish topological vector space V , whenever a cardinality of a family of non-equivalent generators of $\pi(V)$ -shy sets in the entire space is equal to the continuum iff $\dim(V) \geq 2$.

Remark 5.5. Let K be any class of Borel measures defined on a measurable space $(E, \mathcal{B}(E))$. We say that a set $A \in \mathcal{B}(E)$ is absolutely positive with respect to the class K , if

$$(\forall \mu)(\mu \in K \rightarrow \mu(A) > 0).$$

Now, if we denote by K a class of all generators of $\pi(V)$ -shy sets defined in a Polish topological vector space V , then every $\pi(V)$ -prevalent set is absolutely positive with respect to the class K .

The following assertion answers positively to the Problem 5.3.

Theorem 5.5. *Let V be a Polish topological vector space such that $\dim(V) \geq 2$. Then, there exist quasi-finite semi-finite generators of $\pi(V)$ -shy sets (moreover, translation-invariant Borel measures) μ_1 and μ_2 in V , which are not singular or equivalent.*

Proof. Let us consider two linearly independent non-zero elements v_0 and v_1 in V . For $i = 0, 1$, let L_i be a one-dimensional vector subspace defined by v_i and let μ_i be the classical one-dimensional Borel measure in V concentrated on L_i . For $i = 0, 1$, we set $\lambda_i = G_{(\pi(V), \mu_i)}$. It is obvious that λ_1 and λ_2 are translation-invariant Borel measures in V .

By Theorem 3.3, the measure λ_i is a quasi-finite semi-finite generator of $(\pi(V), \mu_i)$ -shy sets in V for $i = 0, 1$. By Theorem 5.4, the measures λ_0 and λ_1 are not singular. We have show that measures λ_0 and λ_1 are not equivalent. Indeed, we have

$$\lambda_0(\{\alpha v_0 : 0 \leq \alpha \leq 1\}) = 1,$$

and

$$\lambda_1(\{\alpha v_0 : 0 \leq \alpha \leq 1\}) = 0,$$

which means that λ_0 and λ_1 are not equivalent. \square

Remark 5.6. Let $(\mathbb{R}^\infty, +)$ be the abelian (additive) Polish group of sequences with its usual product topology. A positive solution of the Problem 5.3 for $(\mathbb{R}^\infty, +)$ can be obtained in another way, if we consider Baker measures constructed in [1] and [2], respectively. The proof of this fact can be found in [27] (see Theorem 15.2.1, p. 204).

6. On a Certain Example of \mathbb{T} -cm and

\mathbb{T} -shy Sets in \mathbb{R}^∞

Let $\alpha = (n_i)_{i \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ such that $n_i = n_j$, for every $i, j \in \mathbb{N}$.

We set $F_i = (a_1^{(i)}, \dots, a_{n_0}^{(i)})$, for every $i \in \mathbb{N}$. Let f be any permutation of

\mathbb{N} such that for every $i \in \mathbb{N}$, there exists $j \in \mathbb{N}$ such that $f(\alpha_k^{(i)}) = \alpha_k^{(j)}$, for $1 \leq k \leq n_0$. Then a map $A_f : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ defined by $A_f((z_k)_{k \in \mathbb{N}}) = (z_{f(k)})_{k \in \mathbb{N}}$ for $(z_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty$, is called a *canonical α -permutations* of \mathbb{R}^∞ .

A group of transformations generated by all α -permutations and shifts of \mathbb{R}^∞ , is denoted by \mathcal{G}_α .

We have the following:

Example 6.1 ([34], Theorem 2, p. 217). For every $\alpha = (n_i)_{i \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ for which $n_i = n_j(i, j \in \mathbb{N})$, the measure ν_α is \mathcal{G}_α -invariant.

Since the measure ν_α is \mathcal{G}_α -invariant and quasi-finite, by the scheme used in [30], we can prove that this measure is the generator of \mathcal{G}_α -shy (equivalently, \mathcal{G}_α -cm) sets in \mathbb{R}^∞ .

7. On a Certain Application of Corollary 3.1

Fremlin [9] posed the following questions.

Question 7.1. Which Banach spaces have the property that there exists a translation-invariant Borel measure μ such that the closed unit ball has measure 1?

Question 7.2. Let ℓ^∞ be a Banach space of all real-valued sequences equipped with norm $\|\cdot\|_\infty$ defined by

$$\|(x_k)_{k \in \mathbb{N}}\|_\infty = \sup_{k \in \mathbb{N}} |x_k|.$$

Let $\mathbb{B}(\ell^\infty)$ be a Borel σ -algebra of subsets of ℓ^∞ generated by the norm $\|\cdot\|_\infty$. Further, let B_0 be a closed unit ball defined by

$$B_0 = \{(x_k)_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in \ell^\infty \text{ \& } \|(x_k)_{k \in \mathbb{N}}\|_\infty \leq 1\}.$$

Does there exist a translation-invariant measure μ defined on the measure space $(\ell^\infty, \mathbb{B}(\ell^\infty))$ such that $\mu(B_0) = 1$?

Lemma 7.1 (Riesz) ([4], Lemma 12.15, p. 483). *Let B be a normed linear space and Y be a proper closed subspace. Then, for every $0 < \delta < 1$, there is an element $x_\delta \in X$ with $\|x_\delta\| = 1$ and such that $\text{dist}(x_\delta, Y) > 1 - \delta$.*

Lemma 7.2. *Let B be an infinite-dimensional Banach space. Then, for every $r > 0$, there exists an infinite sequence of disjoint open balls $(B(x_k, \frac{r}{4}))_{k \in \mathbb{N}}$ of radius $\frac{r}{4}$ and with center at the x_k , which are contained in the ball $\mathbf{B}(0, r)$.*

Proof. Choose an element x_1 with $\|x_1\| = 1$. By applying Lemma 7.1 to the subspace Y_1 spanned by x_1 , there is an element x_2 with $\|x_2\| = 1$ and $\text{dist}(x_2, Y_1) > \frac{1}{2}$. Once again, applying Lemma 7.1 to the subspace Y_2 spanned by $\{x_1, x_2\}$, there is an element x_3 with $\|x_3\| = 1$ and $\text{dist}(x_3, Y_2) > \frac{1}{2}$. Continuing inductively, we obtain a sequence such that each pair of its members is at a distance apart of at least $\frac{1}{2}$. As B is infinite-dimensional, this process cannot stop. Thus, we have an infinite sequence of elements $(x_k)_{k \in \mathbb{N}}$ on the unit sphere. It is obvious that $(B(\frac{x_k}{2}, \frac{1}{4}))_{k \in \mathbb{N}}$, an infinite sequence of disjoint open balls contained in the ball $\mathbf{B}(0, 1)$. Then, $(B(\frac{rx_k}{2}, \frac{r}{4}))_{k \in \mathbb{N}}$ will be an infinite sequence of disjoint open balls contained in the ball $\mathbf{B}(0, r)$. \square

Theorem 7.1. *Let \mathbb{B} be an infinite-dimensional Radon Banach group. Then, there does not exist a translation-invariant Borel measure μ in \mathbb{B} such that the closed unit ball has μ -measure 1.*

Proof. Assume the contrary and let μ be such a translation-invariant Borel measure in \mathbb{B} , which gets a numerical value 1 on the unit closed ball $\mathbf{B}(0, 1)$. Then every closed ball with radius $r < 1$ is a $\pi(B)$ -cm set. Indeed, let $\mathbf{B}(0, r)$ be a closed ball with radius r and center at the zero of the \mathbb{B} . Because the space is infinite dimensional, by Lemma 7.2, there exists an infinite sequence of disjoint open balls $(B(x_k, \frac{r}{4}))_{k \in \mathbb{N}}$ of radius $\frac{r}{4}$, which are contained in the ball $\mathbf{B}(0, r)$. It is clear that $\mu(B(0, \frac{r}{4})) = 0$, which follows from the following condition

$$+\infty \times \mu(B(0, \frac{r}{4})) = \sum_{k \in \mathbb{N}} \mu(B(x_k, \frac{r}{4})) \leq \mu(\mathbf{B}(0, r)) \leq \mu(\mathbf{B}(0, 1)) = 1.$$

Let us show that $\mu(\mathbf{B}(0, r)) = 0$. Assume the contrary and let $\mu(\mathbf{B}(0, r)) > 0$.

Since the restriction $\mu_{\mathbf{B}(0,1)}$ of the μ to the $\mathbf{B}(0, 1)$, defined by

$$(\forall X)(X \in \mathcal{B}(B) \rightarrow \mu_{\mathbf{B}(0,1)}(X) = \mu(B(0, 1) \cap X)),$$

is a Borel probability measure on Radon metric space \mathbb{B} , we claim that there exists a compact set $F \subseteq \mathbf{B}(0, r)$ of non-zero finite $\mu_{\mathbf{B}(0,1)}$ -measure. Hence,

$$0 < \mu_{\mathbf{B}(0,1)}(F) < 1 \Leftrightarrow 0 < \mu(F) < 1.$$

Now, let us consider a covering $(B(0, \frac{r}{4}) + x)_{x \in F}$ of the F . Since F is compact, there exists a finite family $(x_k)_{1 \leq k \leq n}$ of elements of the F such that $F \subseteq \bigcup_{1 \leq k \leq n} B(x_k, \frac{r}{4})$. By using translation-invariance of the μ , we get

$$\mu(F) \leq \mu(\bigcup_{1 \leq k \leq n} B(x_k, \frac{r}{4})) \leq \sum_{1 \leq k \leq n} \mu(B(x_k, \frac{r}{4})) = n\mu(B(0, \frac{r}{4})) = 0.$$

The latter relation is contradiction, and thus, we claim that $\mu(\mathbf{B}(0, r)) = 0$.
Since

$$\mu_{\mathbf{B}(0,1)}(\mathbf{B}(0, r) + h) = \mu((\mathbf{B}(0, r) + h) \cap \mathbf{B}(0, 1)) \leq \mu(\mathbf{B}(0, r) + h) = 0,$$

we claim that the $\mu_{\mathbf{B}(0,1)}$ is a $\pi(B)$ -testing measure to the $\mathbf{B}(0, r)$, which follows that the $\mathbf{B}(0, r)$ is $\pi(B)$ -cm set.

Now, let us show that every closed ball with radius $x > 0$ is $\pi(B)$ -cm set. We define a measure μ_x by

$$(\forall X)(X \in \mathcal{B}(B) \rightarrow \mu_x(X) = \mu\left(\frac{1}{x} X\right)).$$

It is obvious that the μ_x is a translation-invariant Borel measure on \mathbb{B} , which gets a numerical value one on the closed ball of radius x . For $0 < r < x$, we have

$$\mu_x(\mathbf{B}(0, r)) = \mu\left(\mathbf{B}\left(0, \frac{r}{x}\right)\right) = 0,$$

because $\frac{r}{x} < 1$. Using the scheme used above, we claim that every closed ball with radius $r < x$ is $\pi(B)$ -cm set.

Thus, every closed ball in B is $\pi(B)$ -cm set and we get that the B being the countable union of closed balls (equivalently, $\pi(B)$ -cm sets) is not $\pi(B)$ -cm set. The latter relation means that $\pi(B)$ -cm sets does not constitute a σ -ideal and we get a contradiction with Corollary 3.1. \square

As a simple consequence of Theorem 7.1, we have the following assertions.

Corollary 7.1. *Let continuum is not a real-valued measurable cardinal. Then the answer to the Question 7.2 is no.*

Proof. Since the topological weight c of the ℓ^∞ is not a real-valued measurable cardinal, by Ulam's well known result (see [24]), we claim that ℓ^∞ is Radon Banach group. An application of Theorem 7.1 ends the proof of Corollary 7.1. \square

Corollary 7.2 (CH). *The answer to the Question 7.2 is no.*

Proof. Since (CH) is an assertion that $c = \aleph_1$, by Ulam's well known result, we claim that the $c (= \aleph_1)$ is not a real-valued measurable cardinal. An application of Corollary 7.1 ends the proof of Corollary 7.2. \square

Remark 7.1. The positive solution of the Question 7.2 has been constructed in Solovay model (SM) [40] (see, [25]). Since the answer to the Question 7.2 is *no* in the theory ZFC + CH (see Corollaries 7.1 and 7.2), we claim that the Question 7.2 is not solvable within the theory ZF. It is an object of interest to give of a similar result for the Question 7.2.

Note that we can not apply a technique of $\pi(B)$ -cm and $\pi(B)$ -shy sets for such infinite-dimensional Radon Banach spaces, whose cardinality is larger than c (see Section 3). In spite of this phenomena, we can answer negatively to the Question 7.1 for infinite-dimensional Radon Banach spaces in the theory ZFC.

Theorem 7.2. *Let \mathbb{B} be an infinite-dimensional Radon Banach space. Then, there does not exist a translation-invariant Borel measure μ in \mathbb{B} such that the closed unit ball has μ -measure 1.*

Proof. Assume the contrary and let μ be such a translation-invariant Borel measure in \mathbb{B} , which gets a numerical value 1 on the unit closed ball $\mathbf{B}(0, 1)$. By Lemma 7.2, there is an infinite sequence of disjoint open balls $(B(x_k, \frac{1}{4}))_{k \in \mathbb{N}}$ of radius $\frac{1}{4}$, which are contained in the ball $\mathbf{B}(0, 1)$.

Note that $\mu(B(0, \frac{1}{4})) = 0$ because

$$+\infty \times \mu(B(0, \frac{1}{4})) = \sum_{k \in \mathbb{N}} \mu(B(x_k, \frac{1}{4})) \leq \mu(\mathbf{B}(0, 1)) = 1.$$

Since the restriction $\mu_{\mathbf{B}(0,1)}$ of the μ to the $\mathbf{B}(0, 1)$, defined by

$$(\forall X)(X \in \mathcal{B}(B) \rightarrow \mu_{\mathbf{B}(0,1)}(X) = \mu(B(0, 1) \cap X)),$$

is a Borel probability measure on Radon metric space \mathbb{B} , we claim that there exists a compact set $F \subseteq \mathbf{B}(0, 1)$ of non-zero finite $\mu_{\mathbf{B}(0,1)}$ -measure.

Hence,

$$0 < \mu_{\mathbf{B}(0,1)}(F) < 1 \Leftrightarrow 0 < \mu(F) < 1.$$

Now, let us consider a covering $(B(0, \frac{1}{4}) + x)_{x \in F}$ of the F . Since F is compact, there exists a finite family $(x_k)_{1 \leq k \leq n}$ of elements of the F such that $F \subseteq \bigcup_{1 \leq k \leq n} B(x_k, \frac{1}{4})$. By using translation-invariance of the μ , we get

$$0 < \mu(F) \leq \mu(\bigcup_{1 \leq k \leq n} B(x_k, \frac{1}{4})) \leq \sum_{1 \leq k \leq n} \mu(B(x_k, \frac{1}{4})) = n\mu(B(0, \frac{1}{4})) = 0.$$

The latter relation is a contradiction, and thus, Theorem 7.2 is proved. \square

As a simple consequences of Theorem 7.2, we have the following assertions.

Corollary 7.3. *Let B be a Banach space, whose topological weight is not a real-valued measurable cardinal. Then, there does not exist a translation-invariant Borel measure μ in \mathbb{B} such that the closed unit ball has μ -measure 1.*

Corollary 7.4. *Let there does not exist a real-valued measurable cardinal. Then, the answer to the Question 7.1 is no for an arbitrary infinite-dimensional Banach space.*

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